Critical exponents for boundary avalanches in two-dimensional Abelian sandpile

1994 J. Phys. A: Math. Gen. 27 L585

(http://iopscience.iop.org/0305-4470/27/16/004)

J. Phys. A: Math. Gen. 27 (1994) L585–L590. Printed in the UK

LETTER TO THE EDITOR

Critical exponents for boundary avalanches in two-dimensional Abelian sandpile

E V Ivashkevich†, D V Ktitarev‡ and V B Priezzhev†

† Bogolubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980 Russia

‡ Laboratory of Computing Techniques, Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980 Russia

Received 4 July 1994

Abstract. We investigate the properties of boundary avalanches in 2D Abelian sandpile model (ASM). We construct the one-to-one correspondence of boundary avalanches and two-rooted spanning trees. Using the connection between the obtained graph representation and lattice Green functions we calculate the exact values of critical exponents for size and lifetime distributions of avalanches starting at the open boundary that forms an angle α. We find that the probability of a boundary avalanche of the size s varies as $s^{-1-\pi/2\alpha}$ for large s and the probability of an avalanche of lifetime t varies as $t^{-1-4\pi/5\alpha}$ for large t. The obtained values are verified by numerical simulations.

In recent years there have been numerous studies of the sandpile model proposed by Bak *et al* [1,2]. The sandpile models display the mechanism of emergence of power-law spatial and temporal correlations during the evolution of extended dissipative systems. Avalanche processes caused by random perturbations play a crucial role in this self-organized critical (SOC) behaviour. The probability distribution of size and lifetime of avalanches follow power laws $\mathcal{D}(s) \sim s^{-\tau_s}$ and $\mathcal{D}(t) \sim t^{-\tau_t}$. To determine the critical exponents, several simplified models have been suggested by introducing anisotropy [3], directedness [4,5], the Bethe lattice [6] or a complete graph structure [7]. In spite of a variety of methods used in these solutions, the analytical determination of the critical exponents for the basic Abelian sandpile model (ASM) [8] remains an unsolved problem.

In parallel with bulk critical exponents, most of the critical models have non-trivial boundary exponents. If the conformal limit of a given lattice model is known, the explicit relationship between bulk and boundary exponents can be established [9]. In the case of the sandpile model, boundary exponents correspond to avalanches initiated at boundary sites of the lattice. The aim of this letter is to find the exact values of boundary critical exponents τ_s and τ_t for the ASM with open boundary conditions.

We consider the ASM on a $N \times N$ square lattice $\mathcal{L}$. Each boundary site is connected by a bond to the additional site $\star$ which plays the role of the sink. The discrete Laplacian $(N^2+1) \times (N^2+1)$ matrix Δ has non-zero elements Δ_{ii} equal to the number of neighbour sites of i and $\Delta_{ij} = -1$ for all pairs of adjacent sites i and j. The toppling matrix $\Delta^{(\star)}$ is obtained from Δ by deleting the column and row corresponding to $\star$. The height of the sandpile at any site $i \in \mathcal{L}$ takes an integer value z_i. Particles are added at randomly chosen sites and z_i is increased as

$$z_i \to z_i + 1. \qquad (1)$$

If the height z_i exceeds the critical value $\Delta_{ii}^{(*)}$, that site becomes unstable and topples. On toppling at site i

$$z_j \to z_j - \Delta_{ij}^{(*)} \qquad \text{for } j \in \mathcal{L}. \tag{2}$$

All stable configurations of heights have the same probability in the steady state. They are characterized by the absence of forbidden subconfigurations (FSC) on subsets $\mathcal{F} \subset \mathcal{L}$ satisfying the inequalities [8]:

$$z_k \leqslant \sum_{j\in\mathcal{F}} (-\Delta_{kj}^{(*)}) \qquad \text{for all } k \in \mathcal{F}, j \neq k. \tag{3}$$

Dhar [8] has introduced the burning algorithm to determine if a given configuration is allowed in the SOC state and has found the total number of allowed configurations. Based on the burning algorithm, Majumdar and Dhar [11] have established an equivalence between the SOC state of ASM and $q \to 0$ limit of the Potts model that can be represented in turn as a set of spanning trees covering a given lattice. For this reason, any allowed stable configuration can be put into one-to-one correspondence with a spanning tree.

The burning procedure for constructing trees is equivalent to a 'toppling from sink' together with a given order of preference for successive topplings of sites [8]. By this procedure, one adds a particle to each site connected with $\star$ causing all sites of the lattice to topple (otherwise, the configuration would contain an FSC). The spanning tree is a collection of bonds connecting pairs of sites which toppled on successive moments of time. The point $\star$ is the root of the tree $T_\star$.

The translation of the allowed configurations into the language of spanning trees makes it possible to find the fractional numbers of sites having heights 1, 2, 3, 4, [10, 12] and, in general, to get a comprehensive description of the SOC state.

The study of avalanches needs an extension of the tree representation. To this end, we will consider an avalanche process in more detail. The Abelian property admits an arbitrary order of topplings of non-stable sites during an avalanche. We choose a special but quite natural order amongst these. Namely, let us add a particle to the site i having the height Δ_{ii} in an allowed configuration $\mathcal{C}$. We topple it once and then topple all sites that become unstable keeping the site i out of the second toppling. We call the set of sites toppled in this way 'the first wave of topplings'.

After the first wave has gone out, we topple the site i a second time and continue the avalanche not permitting this site to topple a third time. The set of relaxed sites in the period after the first wave is called 'the second wave'. The process continues producing intermediate configurations $\mathcal{C}_1, \mathcal{C}_2, \ldots,$ until the site i undergoes the maximum number of topplings and the avalanche stops.

The cluster of sites toppled in the kth wave forms the subset $\mathcal{F}_k$ of the configuration $\mathcal{C}_k$. If the kth wave is not the last one in a given avalanche, $\mathcal{F}_k$ is FSC. The sites belonging to $\mathcal{F}_k$ topple during the kth wave only once. Indeed, let us assume that a certain site j has toppled the first time after its neighbour j'. Then, j would have the second toppling only after topplings at all its neighbours, including j'. Therefore, to topple j twice, we have first to topple j' twice. As the initial toppling at site i never repeats during the given wave, the other sites of $\mathcal{F}_k$ topple once, as well.

The procedure, which is inverse of that described, has been introduced recently by Dhar and Manna [13].

To find the tree representation of waves, we consider the ASM model on an auxiliary lattice $\mathcal{L}'$, consisting of the original lattice $\mathcal{L}$, the site $\bullet$ connecting with boundary sites of

$\mathcal{L}$ and an additional bond connecting the site $\star$ and a given site i situated inside the lattice. Accordingly, we change the element of the topping matrix $\Delta_{ii}^{(\star)} \rightarrow \Delta_{ii}^{(\star)} + 1$. Applying the burning procedure, we construct the set of spanning trees on the lattice $\mathcal{L}'$. All trees can be divided into two classes. The first class consists of trees not containing the bond $(\star i)$ and therefore coincides with the set of one-rooted spanning trees $T_\star$ on the original lattice. The trees of the second class contain the bond $(\star i)$. On deleting the bond $(\star i)$ a subtree T_i gets disconnected. Considering the site i as a root of T_i, we obtain a two-rooted situation where a spanning tree on the original lattice consists of two disconnected components $T_\star$ and T_i.

According to the burning procedure, the spanning trees are obtained by adding particles to all neighbouring sites of $\star$, including i. The particle added to i can be considered as a perturbation giving rise to an avalanche. Since i is connected with $\star$, it topples only once and the avalanche is actually the wave. This wave corresponds to the subtree T_i. Also, one can first add $k-1$ particles to i and then apply 'the toppling from the sink'. On the one hand, this process produces the kth wave of topplings and, on the other hand, it leads to the subtree corresponding to this wave.

On the contrary, given a two-rooted tree, one can reconstruct a unique configuration of heights using the order of preference of the burning procedure [11].

Thus, in addition to the correspondence between allowed stable configurations and one-rooted spanning trees $\{T_\star\}$, we get the one-to-one correspondence between all waves of topplings and two-rooted spanning trees $\{T_\star \cup T_i\}$.

The graph representation of waves enable us to link the toppling process and the lattice Green function G_{ij}. For this purpose, we shall prove the following

Proposition. For an arbitrary connected graph Γ with a fixed vertex $\star$,

$$G_{ij} = \mathcal{N}^{(i,j)}/\mathcal{N} \tag{4}$$

where $\mathcal{N}^{(i,j)}$ is the number of two-rooted spanning trees having the roots $\star$ and i, such that both vertices i and j belong to the same one-rooted subtree; $\mathcal{N}$ is the total number of spanning trees on Γ.

Proof. Let Δ be the symmetrical Laplacian matrix of the graph Γ. The Kirchhoff's matrix theorem [14] reads

$$\mathcal{N} = \det \Delta^{(\star)} \tag{5}$$

where matrix $\Delta^{(\star)}$ is obtained from Δ by deleting the column and row corresponding to the root $\star$. By the Kirchhoff's formula for resistance, the number $\mathcal{N}^{(i)(j)}$ of two-component spanning trees having the vertices i and j in *different* components is

$$\mathcal{N}^{(i)(j)} = \det \Delta^{(i)(j)} \tag{6}$$

where the matrix $\Delta^{(i)(j)}$ is obtained from $\Delta^{(\star)}$ by deleting the columns and rows corresponding to the vertices i and j.

Instead of deleting elements of $\Delta^{(\star)}$, we can add ϵ to the elements $\Delta_{ii}^{(\star)}$ and $\Delta_{jj}^{(\star)}$ and $-\epsilon$ to the elements $\Delta_{ij}^{(\star)}$ and $\Delta_{ji}^{(\star)}$ obtaining the new matrix $\Delta_\epsilon^{(\star)}$. Then

$$\mathcal{N}^{(i)(j)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \det \Delta_\epsilon^{(\star)} \tag{7}$$

and we can evaluate the ratio $\mathcal{N}^{(i,j)}/\mathcal{N}$ by the formula

$$\frac{\det \Delta_{\epsilon}^{(\star)}}{\det \Delta^{(\star)}} = \det(I + GB) \tag{8}$$

where $B = \Delta_{\epsilon}^{(\star)} - \Delta^{(\star)}$ and $G = (\Delta^{(\star)})^{-1}$ is the solution of the Poisson equation with the boundary conditions $G_{\star k} = 0$ for all k. Direct evaluation in (8) leads to

$$\mathcal{N}^{(i)(j)}/\mathcal{N} = G_{ii} + G_{jj} - G_{ij} - G_{ji}. \tag{9}$$

Putting $i = \star$ we also have

$$\mathcal{N}^{(\star)(j)}/\mathcal{N} = G_{jj} \qquad \text{for all } j \neq \star. \tag{10}$$

The number $\mathcal{N}^{(\star)(j)}$ is the sum of two parts

$$\mathcal{N}^{(\star)(j)} = \mathcal{N}^{(\star j)(j)} + \mathcal{N}^{(\star)(ij)} \tag{11}$$

where the notation (ij) means filling both vertices into one component. Analogously

$$\mathcal{N}^{(i)(j)} = \mathcal{N}^{(\star i)(j)} + \mathcal{N}^{(i)(\star j)}. \tag{12}$$

Since, by the definition, $\mathcal{N}^{(\star)(ij)} = \mathcal{N}^{(ij)}$, (4) is a simple consequence of the linear equations (9–12).

Due to the relationship between two-rooted trees and waves, we conclude that $\mathcal{N}G_{ij}$ is the number of waves initiated at the site i and involving the site j.

The derived result is in agreement with the observation by Dhar [8] that G_{ij} is the expected number n_{ij} of topplings at the site j due to the avalanches caused by adding a particle at i. Indeed, as each wave corresponds to exactly one toppling of all its sites, n_{ij} coincides with the expected number of waves involving the site j.

Now, we can use the Green function representation for finding the wave distributions. First, we consider the waves deep inside the lattice without reference to particular avalanches to which every wave belongs. Assuming isotropy and compactness of waves, we can estimate the probability $\mathcal{P}(r \geqslant r_{ij})$ that the radius of the wave is not less than the distance between i and j as

$$\mathcal{P}(r \geqslant r_{ij}) \sim G_{ij}. \tag{13}$$

The size of avalanches scales as $s \sim r^2$. Then the wave probability distribution $\mathcal{D}(s) \sim 1/s$ follows immediately from the known asymptotics of the Green function $G(r) \sim \ln r$.

Returning to the boundary avalanches, we note that in this case each avalanche consists of only one wave. The reason is that any boundary site i has the root $\star$ as the neighbour site. Therefore, the second toppling of the site i is impossible because of the lack of topplings at $\star$. The asymptotic form of the boundary Green function in the continuum limit is

$$G(r) \sim \log|\boldsymbol{r} - \boldsymbol{a}| - \log|\boldsymbol{r} + \boldsymbol{a}| \sim \frac{(\boldsymbol{a}, \boldsymbol{r})}{r^2} \tag{14}$$

where $\boldsymbol{a}$ is a unit vector perpendicular to the boundary. Correspondingly, the probabilities that the front of the avalanche exceed r is

$$\text{prob}(r' > r) \sim \frac{1}{r} \tag{15}$$

which leads, after differentiation, to the radius distribution $\mathcal{D}(r) \sim 1/r^2$. Using, as before, the relations $s \sim r^2$ and $\mathcal{D}(s)\,\mathrm{d}s \sim \mathcal{D}(r)\,\mathrm{d}r$, we get the sought probability distribution

$$\mathcal{D}(s) \sim \frac{1}{s^{3/2}}. \tag{16}$$

The relationship between the exponents τ_s and τ_t can be found from the following arguments. By the construction of trees, the avalanche process follows the branch structure of the tree. Then, the duration of avalanche T varies as the chemical distance l of the tree [11]. The exact asymptotics $I \sim r^{5/4}$ is known for the q-component Potts model at $q = 0$ [15]. This implies that

$$5\tau_t = 8\tau_s - 3. \tag{17}$$

Thus, for the boundary avalanches, we have $\tau_s = 3/2$ and $\tau_t = 9/5$.

The above arguments can be easily generalized to the boundaries forming an arbitrary angle α. It is known from the theory of complex variables that the Green function of the Laplacian in the region bounded by angle α has the form

$$G(x, y) = \frac{1}{2\pi}\,\mathrm{Im}(z^{-\pi/\alpha}) \tag{18}$$

where $z = x + \mathrm{i}y$, (x, y) are the Cartesian coordinates. Then, the function $G(r)$ decays as

$$G(r) \sim r^{-\pi/\alpha} \tag{19}$$

for any direction apart from arms of the angle. This leads to the distribution

$$\mathcal{D}(r) \sim r^{-1-\pi/\alpha}. \tag{20}$$

Again using the relation $s \sim r^2$, we get the asymptotics

$$\mathcal{D}(s) \sim s^{-1-\pi/2\alpha} \tag{21}$$

which corresponds to $\tau_s = 1 + \frac{\pi}{2\alpha}$ and $\pi_t = 1 + \frac{4\pi}{5\alpha}$.

The analytical results have been verified numerically by Monte Carlo simulations. We considered lattices with sizes up to 100 having angles $\alpha = \pi/2, \pi, 3\pi/2, 2\pi$ with statistics up to 10^6 avalanches.

Table 1. Angular critical exponents for multiples of $\pi/2$.

α	$\pi/2$	π	$3\pi/2$	2π
τ_s	1.9	1.51	1.32	1.21
exact	2	3/2	4/3	5/4

Table 1 shows that the numerically determined values are in good agreement with our predictions.

The angle 2π is of special interest. In this case, avalanches start at the top of a cut of the plane. The geometry of the avalanches closely resembles the one occurring deep inside the lattice. So, one can expect that the critical exponents in both the cases are in close agreement. Indeed, the difference between numerical estimations by Manna [16] $\tau_s = 1.22$ and $\tau_t = 1.38$ and the boundary exponents near the cut $\tau_s = 1.25$ and $\tau_t = 1.40$ is not more than 3%.

One of us (VBP) thanks Dublin Institute for Advanced Studies for kind hospitality.

References

[1] Bak P, Tang C and Wiesenfeld K 1987 *Phys. Rev. Lett.* **59** 381
[2] Bak P, Tang C and Wiesenfeld K 1988 *Phys. Rev.* A **38** 364
[3] Hwa T and Kardar M 1989 *Phys. Rev. Lett.* **62** 1813
[4] Kadanoff L P, Nagel S R, Wu L and Zhou S M 1989 *Phys. Rev.* A **39** 6524
[5] Dhar D and Ramaswamy R 1989 *Phys. Rev. Lett.* **63** 1659
[6] Dhar D and Majumdar S N 1990 *J. Phys. A: Math. Gen.* **23** 433
[7] Janowsky S A and Laberge C A 1993 *J. Phys. A: Math. Gen.* **26** L973
[8] Dhar D 1990 *Phys. Rev. Lett.* **64** 1613
[9] Cardy J L 1987 *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J L Lebowitz (London: Academic) p 55
[10] Majumdar S N and Dhar D 1991 *J. Phys. A: Math. Gen.* **24** L357
[11] Majumdar S N and Dhar D 1992 *Physica* **185A** 129
[12] Priezzhev V B 1994 *J. Stat. Phys.* **74** 955
[13] Dhar D and Manna S S 1994 *Phys. Rev.* E 49 2684
[14] Harary F 1990 *Graph Theory* (Reading, MA: Addison Wesley) p 192
[15] Coniglio A 1989 *Phys. Rev. Lett.* **62** 3054
[16] Manna S S 1990 *J. Stat. Phys.* **59** 509